PHY180 Final Report
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1 Introduction

The purpose of this report was to determine whether or not a simple home-made pendulum follows the imposed upon mathematical model(s). To begin, a period vs angle experiment was conducted. During this experiment, the data outputted a best fit line of a quadratic function following the formula

$$T = T_0(1 + B\theta_0 + C\theta_0^2) \tag{1}$$

the outputted best fit curve and data points can be seen in **Figure 2**, the data extracted from this figure showed that $B = 0.00 \pm 0.07$ (experimentally 0) and that $C = 0.36 \pm 0.07$ (experimentally not 0). In **Equation 1**, B's value relates to symmetry and C's value relates to period depending on angle. The imposed upon statement said that angle and period are independent of each other, however the provided data shows that C is not experimentally 0, which means that the claim made by the instructor is false.

To find the angle range where the small-angle approximation $\sin\theta \approx \theta$ is valid, the predicted period $T(\theta)$ was compared to the best-fit value T_0 from Eq. (1). If the predicted period stayed within one standard deviation of T_0 (that is, within $T_0 \pm 0.07\,\text{s}$), the angle was considered to be in the "small-angle" region. Using this method, the approximation was found to hold up to about $|\theta| \lesssim 0.24\,\text{rad}$ ($\approx 15°$). To satisfy, all future experiments were released at 15°.

Afterwards an experiment was done on the decay behavior of the pendulum, showing that it did follow the damped harmonic oscillator equation, which is defined as

$$\theta(t) = \theta_0 e^{-t/\tau}\cos(2\pi t/T + \phi_0) \tag{2}$$

The figure of the data can be seen in **Figure 3**. Afterwards, the peaks were extracted and modeled using a best fit exponential decay function, which followed

$$\theta(t) = \theta_0 e^{-t/\tau} \tag{3}$$

This outputted the best fit line which can be seen in **Figure 4** with the equation of:

$$\theta = 0.526244 \times e^{-t/33.35\pm0.02} - 0.017733 \tag{4}$$

It was through this that τ, the decay constant, was defined as 33.35s, allowing for the Quality Factor (Q Factor) to be calculated.

The Q factor is the number of oscillations it takes for the pendulum's amplitude (angle) to decay to $e^{-\pi}$, or $\approx 4\%$. The first method of deriving it is using the equation

$$Q = \pi\frac{\tau}{T} \tag{5}$$

Using this method, the Q factor was determined to be 96 ± 6. The second method was manually checking how many oscillations it takes for the pendulum's amplitude (period) to decay to 4%, this was done experimentally counting and gave a Q Factor of 97 ± 1. These two values are experimentally consistent, proving that the setup for the pendulum and data extracted was mostly accurate.

Afterwards, the relationship between period and length was investigated. It was imposed that the function of period vs. length should fit a power law of

$$T = kL^n \tag{6}$$

where $k = 2$ and $n = 0.5$. However it must first be accounted for that earlier on, the instructor approximated T as $\approx 2\sqrt{L}$. The real equation however is

$$T = 2\pi\sqrt{\frac{L}{g}},$$

the instructor reached this approximation due to the fact that $\frac{\pi}{\sqrt{g}} \approx 1$. Throughout the experiment conducted, it was found that the pendulum follows the prediction of $k = 2$ and $n = 0.5$ when using the actual formula, not the approximated one (within uncertainties). Throughout the experiment it was determined that there is a direct relationship between length and period that fits a power law function (see **Figure 5**). It was proven that this is truly a power law function after taking the log of both sides, since the graph outputted was linear (see **Figure 6**). This proves its a power function fit as a log-log graph will only produce a linear graph if the original function was a power function of the form $y = kx^n$.

Experimentally, the best–fit parameters obtained were $k = 2.09 \pm 0.07$ and $n = 0.47 \pm 0.05$, which agree with the predicted values within uncertainty. These numerical values confirm that the measured pendulum behaviour follows the expected power–law trend.

Finally, the relationship between the Q Factor and length was measured, the Q Factor was measured at 10 different lengths with 0.1m interval jumps. It was determined earlier that both methods for calculating the Q Factor yield the same result within uncertainties, therefore using either method would've been valid. For this experiment, Method Two was used because it directly counts oscillations until the amplitude reaches $e^{-\pi}$ and avoids model–dependent error. It was determined that as length increases, Q follows a negative power law fit, (see **Figure 7**).

Quantitatively, the Q factor decreased from approximately $Q \approx 140$ at $L = 0.10\,\text{m}$ to about $Q \approx 45$ at $L = 1.00\,\text{m}$, showing a strong length–dependent damping behaviour. These measured values support the negative power–law trend obtained from the fit.

2 Methods and Procedures

The pendulum was constructed using wood bits drilled together to ensure a stabilized frame and to minimize unwanted movements such as oscillations, or elliptical paths caused by the structure. A safety ball (the tennis-ball shaped one put on chairs) was hot glued to a string, the string was wrapped around the top of the pendulum. The mass of the bob was measured to be $0.25\,\text{kg} \pm 0.002\,\text{kg}$. On the other hand, the string's mass was measured to be $0.015\,\text{kg} \pm 0.002\,\text{kg}$. The uncertainty comes from the type B uncertainty specified by the manufacturer of the scale. Through this we can see that the string is relatively massless compared to the bob. For consistency, the trials from Sections 3.1 to 3.3 used a string length of 30cm (0.3m) with a uncertainty of ± 0.005m, the uncertainty was derived from **Reference 1**. The length measured in the previously stated sections started from the start of the string to center of mass of the ball. After performing the Period vs. Angle part of the experiment (which required multiple angles to be tested), all measurements afterwards were conducted with a release angle of $15°$ ($\pi/12$ rad) to maintain stabilized conditions and reduce variability/unneeded uncertainties in the results. These choices helped ensure that the pendulum moved in 2D as much as possible, making the results reproducible and more accurate. It was observed that the pendulum was uneven at the bottom, to fix this the pendulum itself was drilled to a surface, after doing this previously observed elliptical motion went away and the experiment(s) became much more accurate.

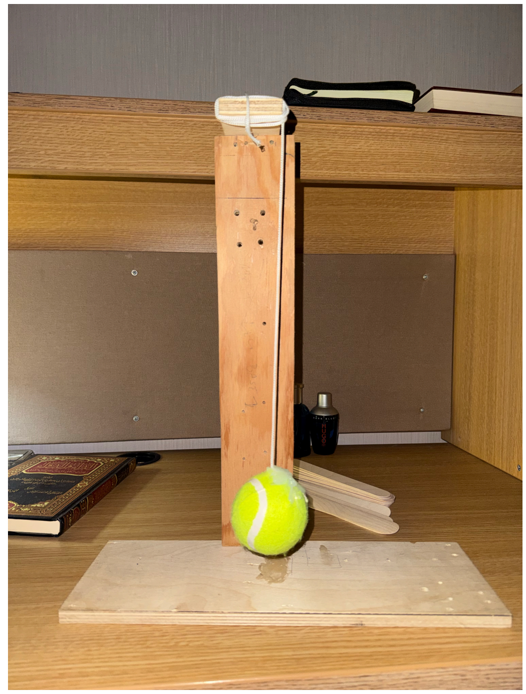

Figure 1: Setup of the pendulum constructed in MyFab. The figure shown has $L = 0.3$m. Two wooden pieces were drilled together and then a third piece was drilled to the top for the first to be wrapped around. The mass of bob is approximately 0.25kg, and the string is relatively massless.

3 Results and Data Analysis

3.1 Period vs. Angle

The period was measured at different release angles (θ) ranging from $-\pi/3$ to $\pi/3$. θ was measured using a physical protractor. The Type B uncertainty in θ was $\pm 0.5°$ (± 0.009 rad), the uncertainty of a protractor. The period (T) was measured using a stopwatch, and divided by the number of oscillations to reduce uncertainty. The Type B uncertainty in T arises from human reaction time, estimated as ± 0.25s (see **Reference 3**) per measurement, which reduces to ± 0.07s after averaging over 3 oscillations. Only the highest uncertainty is considered, that being the one of human reaction time.

Using mathematical model's provided by **Reference 1**, said models were to plot all points, uncertainties, and the line of best fit. The model outputted the quadratic fit in **Equation 1**. See **Figure 2**

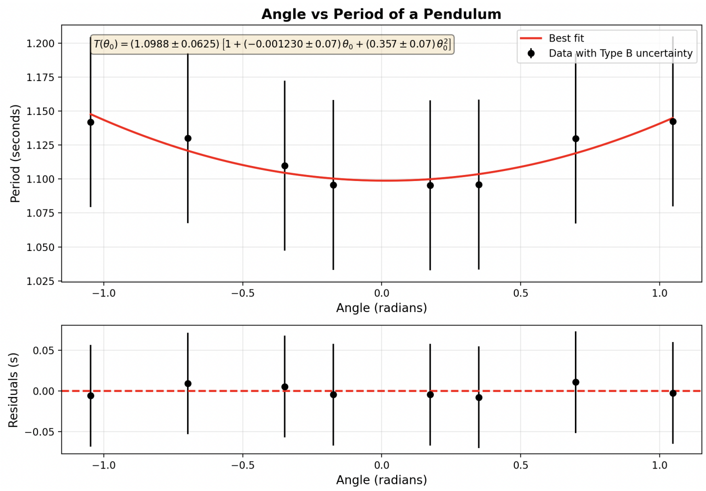

Figure 2: Period vs. Angle including uncertainties. The parabolic fit gives $T_0 = 1.10 \pm 0.07$ s, $B = 0.00 \pm 0.07$, and $C = 0.36 \pm 0.07$. The residuals show that the actual data presented was very close to the expected fit.

3.2 Q Factor (Method One:)

The first method implemented to find the Q factor used the formula in **Equation 5** where τ is the decay constant and T is the period given by the formula $T = 2\pi\sqrt{\frac{L}{g}}$. When solving for T, it must be noted that has a type B uncertainty of 0.5cm (0.005m), which is t uncertainty of the ruler. Knowing this, the period can solved for as: $T = 2\pi \times \sqrt{0.3 \pm 0.005/9.8} = 1.10 \pm 0.$ To find the exponential decay function and to find τ tracker software (see **Reference 2**) was used to au track ≈ 2300 data points of the pendulum decaying o a span of $\approx$ 80 seconds. The x-positions from th data points were converted to angles in radians usi the equation.

$$\theta = \arcsin\left(\frac{x}{L}\right),$$

where L is the length of the string (0.3m), and x the specific x position it was in that data point. T equation was plugged into a mathematical model usi that converted all the data points into radians from x-position. All these converted points were then used to plot a angle-time graph which can be seen in **Figure 3** and followed the damped harmonic oscillator equation, defined in **Equation 2** Afterwards, a model was used to extract all the peaks from **Figure 3** and found the best fit line for the exponential decay function which can be seen in **Equation 4**.

The extracted data and the best fit line can be seen in **Figure 4**. From this, it can be seen that τ is defined as 33.35 ± 0.02 s. The uncertainty of τ was given by the type B uncertainty of the tracker, with the video being recorded on 30 FPS. Therefore the uncertainty would be $1/(30 \cdot 2) = 0.02$. It was already referenced that $T = 1.10 \pm 0.07$, which means **Equation 5** can be solved now that all the unknowns are present, demonstrated below:

$$Q_1 = \pi\frac{33.35 \pm 0.02}{1.10 \pm 0.07} = 96.02 \pm 6$$

This means that **the Q factor (for method one) is defined as:** 96 ± 6

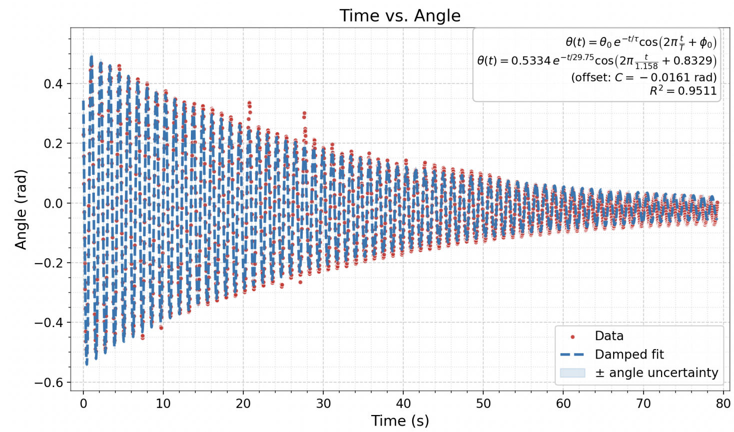

Figure 3: Angle vs. Time graph with consisting of over 2300 data points of the damped harmonic oscillator equation, defined in **Equation 2**. The blue line is the best fit line and the red dots are the all the data points.

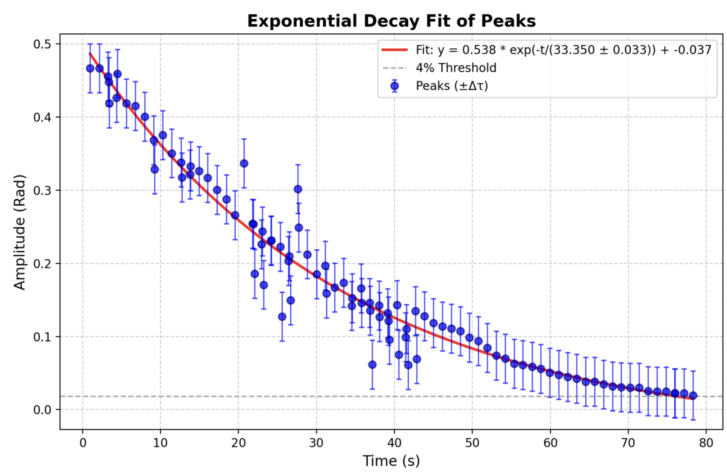

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Figure 4: Best fit exponential decay function with uncertainties, the red line is the line of best fit and the blue points are the data points, the grey threshold is the point where it decays to $e^{-\pi}$ ($\approx 4\%$)

3.3 Q Factor (Method Two)

The second method to determine the Q factor would usually consist of counting the number of oscillations that it takes to decay to $e^{-\pi}$ ($\approx 4\%$) of the initial amplitude. However, another method was employed. If you reference back to **Figure 4**, it can be seen that the same mathematical model that outputted the decay function best-fit line also had a $e^{-\pi}$ threshold. It was determined that it took 97 peaks (or in other words, oscillations) to decay to this level. The uncertainty is given by the uncertainty of counting (± 1, **Reference 1**).

$$Q_2 = 97 \pm 1.$$

This means that **the Q factor (for method two) is defined as:** 97 ± 1

3.4 Length vs Period

The pendulum period T was measured for ten lengths L = 0.1–1.0 m in 0.1 m steps, with a consistent 15° release to provide sufficient initial energy while reducing uncertainty. The uncertainties used throughout were the type-B length uncertainty of ± 0.005 m (ruler) and the uncertainty of the period, defined as human reaction time (0.25s) (see **Reference 3**) divided by the three oscillations timed which gives ± 0.07s. Using the data compiled, a graph was generated of the plots of all the data points measured, including a log–log analysis to check whether the best-fit function will act as a power-function, see **Figure 6**. The fitted parameters produced are $k = 2.09 \pm 0.07$, and $n = 0.47 \pm 0.05$ as can be seen in **Figure 5**

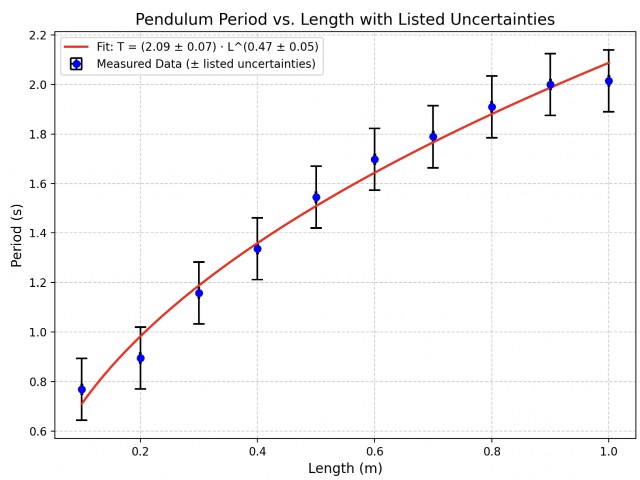

Figure 5: Length vs. Period graph measured from 0.1m to 1m with 0.1m interval jumps. This equation gave the outputs as $k = 2.09 \pm 0.07$ and $n = 0.47 \pm 0.05$.

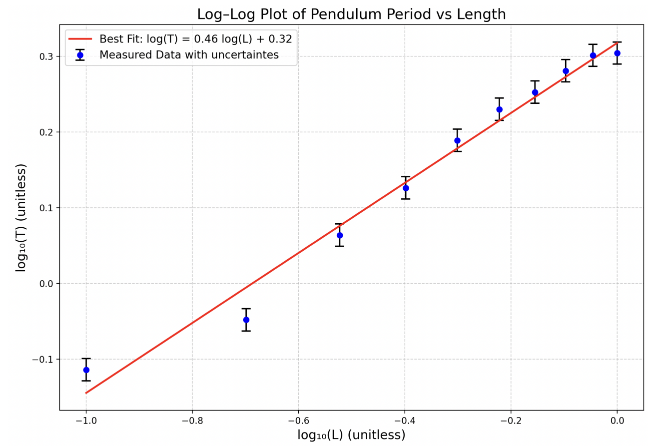

Figure 6: Log–log plot of $\log_{10} T$ versus $\log_{10} L$. The observed trend supports a power–function relationship, with the slope giving n and the intercept giving $\log_{10} k$.

3.5 Length vs Q Factor

While measuring the period for each length, the Q factor was measured using Method 2. For each trial, the number of oscillations required for the amplitude to decay to approximately $\approx 4\%$ were counted. The only uncertainties used here are the type-B uncertainty of counting for Q, given as ± 1 in Q and the type-B ruler uncertainty of ± 0.005m for L. Several candidate models (linear, exponential, and power–law) were fit to the Q–versus–L data. To decide which model was most appropriate, the uncertainties in the fitted parameters were compared for each case. The inverse power–law model consistently produced the smallest parameter uncertainties and the highest goodness-of-fit, while the alternatives showed larger uncertainties and systematic structure in their residuals. This quantitative comparison confirmed that the negative power–law relationship used in Figure 7 is the most reliable description of how Q depends on the pendulum length. The resulting Q–L data and the corresponding fit are shown below.

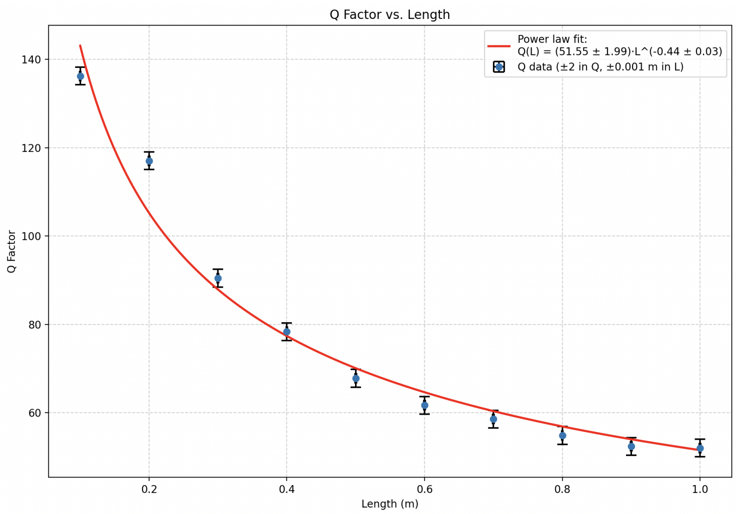

Figure 7: The data points of every Q vs respective length measured, the best fit line outputted a negative power function, showing that there is a direct relationship between length and Q, and one that is evidently very strong

4 Conclusion

The objective was to study the effect of release angle on the pendulum period and to determine the quality factor Q using two independent methods. For the angle period graph, it was determined using **Equation 1** and its values it can be seen that $B = 0.00 \pm 0.07$, this means it is experimentally 0 as the decimal places of B should only be measured to be the decimal places of its uncertainty. On the other hand it's observed that $C = 0.36 \pm 0.07$ does not follow this rule, which means it is not experimentally 0. This means that the initial proposed statement of angle and period being independent is proven false by the conducted experiment.

The Q factor determined from Method 1 was $Q_1 = 96 \pm 6$, while Method 2 gave $Q_2 = 97 \pm 1$. Since Method 2 counts the number of oscillations until the amplitude reaches 4%, the uncertainty is dominated solely by counting error (± 1). This gives $Q_2 = 97$ ± 1. It can be seen that Q_1 and Q_2 are equal within uncertainties. Originally, this was not the case but after re-taking calculations and reducing uncertainties the results became more accurate. Uncertainties were reduced by re-taking data, taking multiple measurements, removing "odd" points, reducing human error during release and stabilizing the constructed structure more.

Afterwards, length vs. period was investigated, the goal was to compare the results to the predictions imposed by the instructor where $k = 2$ and $n = 0.5$, within their respective uncertainties. From the data, it was obtained that $k = 2.09 \pm 0.07$ and $n = 0.47 \pm 0.05$ (see **Figure 5**). The figure shows that n agrees with the prediction, while k does not. At least initially, it must be taken into account however the approximation of $T \approx 2\sqrt{L}$ made by the instructor, this approximation was derived by approximating $\sqrt{g}$ as π. Although this approximation is "technically" true when rounding, the measured k differs from the predicted value by just 0.02 of the imposed predicted value. When this small approximation is taken into account, k is effectively consistent with the prediction.

Lastly, the relationship between the Q factor and length were measured. The objective was to test for a correlation between the quality factor Q and the length of the pendulum L. In **Figure 7**, the data follow a clear negative power law function. This indicates that Q decreases as L increases.
It should be noted that for this report, the data points in **Figure 4** was revised and Method One was refined. Previously, several data points fell noticeably outside the trend suggested by the best-fit line, which skewed the estimation. After re-taking those measurements, the resulting value of τ shifted accordingly, bringing the calculated Q_1 into agreement with Q_2 within experimental uncertainty.

5 Uncertainties

Table 1: Summary of uncertainties in the entire report.

Quantity	Value	Uncertainty
Release angle θ	$-\pi/3$ to $\pi/3$	$\pm 0.5^\circ$
Period T (angle study)	1.09–1.14 s	± 0.07 s
Parabolic fit T_0	1.09 s	± 0.07 s
Parabolic fit B	0 (exp.)	± 0.07
Parabolic fit C	0.36	± 0.07
String length L	0.30 m	± 0.005 m
Tracker frame-time	1/30 s	± 0.02 s
Decay constant τ	33.35 s	± 0.02 s
Period T (Eq. 5)	1.10 s	± 0.07 s
Q factor (Method 1)	96	± 6
Q factor (Method 2)	97	± 1
k (Eq. 6)	2.09	± 0.07
n (Eq. 6)	0.47	± 0.05

It can be seen through this table that the largest uncertainty was the uncertainty provided for calculation for Q_1, to further decrease this uncertainty next time, the best method would be to take multiple sets of data when calculating the Q factor and average it out to reduce uncertainty. Additionally, the uncertainty relating to FPS and the decay function calculation should be noted, the best way to reduce it would be to take a higher quality video, and to take multiple data sets. The uncertainties in this report have been reduced significantly over the course of the experiment, and if it were to be quantified it would be said that we are confident in this model.

6 References

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